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INTEGRAL EQUATION OF THE INVERSE PROBLEM OF THE PLANE THEORY OF ELASTICITY

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Determination of the form of equal strength hole contours in a perforated plane under a specified load, i.e. the inverse problem, was formulated and solved with sufficient degree of generality by Cherepanov [1, 2] who reduced it to the Dirichlet problem for the exterior of a system of parallel slits in a plane, in the class of functions with power singularities at the ends of the slits, and a closed solution was obtained in a number of cases. In the present paper the initial problem for an arbitrary, finitely connected region is reduced to a Fredholm-type equation relative to the density of integral representation of the function which maps conformally a plane with circles excluded, onto a plane of the same connectivity with an unknown boundary. The equation obtained is solved by the method of least squares and this leads, in the case of any finitely connected region, to an unique computational scheme which can be programed into a computer. The coefficients of the corresponding algebraic system are determined and a one-parameter family of the contours sought is constructed for a plane, symmetrically periodic distribution of holes, as an example.

As we know [3], a canonical domain obtained from the ζ-plane by removing n circles, can be mapped onto any n-connected domain S_+ of the complex z-plane with a point at infinity. When $n > 2$, the mapping $\omega_0(\zeta)$ which has the form $\omega_0(\zeta) = C\zeta + \omega(\zeta)$, where $\omega(\zeta)$ is bounded at infinity, depends on $3n$ real parameters, six of which (e.g. one circumference, one fixed point on this circumference and a center of another circumference) can be specified arbitrarily, and C is a scale multiplier. Consequently, a system of contours of equal strength, if it exists, forms a $(3n-6)$-parameter family. The limits of variation of the parameters can be found from geometrical considerations. The presence of symmetry may lead to reduction in the number of parameters.

We have the following relations [4] for determining the stress components at the boundary Γ of the region S_+:

$$\sigma_r + \sigma_\theta = 4\operatorname{Re}\Phi_0(\xi) \tag{1}$$

$$\sigma_\theta - \sigma_r + 2i\tau_{r\theta} = \frac{2(\xi - a_k)^2}{r_k^2\overline{\omega'(\xi)}}\left(\overline{\omega_0(\xi)}\,\Phi_0'(\xi) + \omega_0'(\xi)\,\Psi_0(\xi)\right) \tag{2}$$

Here σ_r, σ_θ and $\tau_{r\theta}$ denote the normal and shear stresses in a polar coordinate system with a pole at the center a_k of a circle of radius r_k and a boundary Γ_k, $k = 1, 2, \ldots, n$. If a homogeneous state of stress with the stress components σ_x, σ_y and τ_{xy} is given at infinity, then $\Phi_0(\zeta)$ and $\Psi_0(\zeta)$ have the form [4]

$$\Phi_0(\zeta) = {}^1/_4(\sigma_x + \sigma_y) + \Phi(\zeta)$$
$$\Psi_0(\zeta) = {}^1/_2(\sigma_y - \sigma_x) + i\tau_{xy} + \Psi(\zeta)$$

where $\Phi(\zeta)$ and $\Psi(\zeta)$ are holomorphic in S_+ and have an asymptotics of the order $O(z^{-2})$ when $z \to \infty$. Assuming that $\sigma_r = p$ at all contours, we set $\sigma_\theta = \sigma_x + \sigma_y - p$ to find, that in this case (1) has a solution [5]

$$\Phi_0(\zeta) = {}^1/_4(\sigma_x + \sigma_y)$$

and (2) reduces to the relation

$$\frac{Car_k^2}{(\xi - \alpha_k)^2} + \frac{\overline{ar_k^2\omega'(\xi)}}{(\xi - a_k)^2} = Cb + \omega'(\xi)\Psi(\xi) \tag{3}$$
$$a = {}^1/_2(\sigma_x + \sigma_y) - p + i\tau, \quad b = {}^1/_2(\sigma_y - \sigma_x) + i\tau_{xy}$$

Let us now turn our attention to the second term in the left-hand side of (3). Taking into account the fact that $r_k^2/(\xi - a_k)^2 = -d\bar{\xi}/d\xi$ when $\xi \in \Gamma_k$, we can write this term in the form

$$\frac{r_k^2\overline{\omega'(\xi)}}{(\xi - a_k)^2} = -\lim_{\Delta\xi\to 0}\frac{\overline{\Delta\xi}}{\Delta\xi}\;\frac{\overline{\omega(\xi+\Delta\xi)} - \overline{\omega(\xi)}}{\overline{\Delta\xi}} = -\frac{d}{d\xi}\overline{\omega(\xi)} \tag{4}$$

Substituting (4) into (3) and integrating, we obtain

$$F(\xi) + \overline{\omega(\xi)} = -C\frac{b}{a}\xi - C\frac{r_k^2}{\xi - a_k} + d_k \tag{5}$$
$$\xi \in \Gamma_k, \quad k = 1, 2, \ldots, n$$

The function $F(\zeta)$ is holomorphic in S_+, $F'(\zeta) = \omega'(\zeta)\Psi(\zeta)/a$ and d_k are arbitrary constants. Varying d_k, if necessary, we can reach the state when the bounded functions $F(\zeta)$ and $\omega(\zeta)$ decrease at infinity. To solve the boundary value problem (5) we shall write, following Sherman [6], $F(\zeta)$ and $\omega(\zeta)$ in terms of the Cauchy-type integrals

$$F(\zeta) = \frac{1}{2\pi i}\int_\Gamma \frac{u(t)}{t - \zeta}dt, \quad \omega(\zeta) = \frac{1}{2\pi i}\int_\Gamma \frac{\overline{u(t)}}{t - \zeta}dt \tag{6}$$

where $\mu(t)$ is a smooth complex-valued function on Γ. The expressions (6) satisfy the condition of decreasing at infinity. Substituting (6) into (5), we obtain

$$u(\xi) + \frac{1}{2\pi i}\int_\Gamma u(t)\,d\ln\frac{t - \xi}{\bar{t} - \bar{\xi}} - d_k = -C\frac{b}{a}\xi - \frac{Cr_k^2}{\xi - a_k} \tag{7}$$

the constants d_k can be determined from the relation

$$d_k = -\frac{1}{2\pi r_k}\int_{\Gamma_k} u(t)\,dS, \quad dS = |dt| \tag{8}$$

Equation (7) represents a Fredholm-type equation with a real, symmetric kernel. Separating the real and imaginary parts we obtain a pair of integral equations relative to potentials $(u(t)$ and $v(t)$ of a double layer, of a modified [7] Dirichlet problem in the class of bounded functions continuous in S_+. (In [2] a dual Dirichlet problem was solved in a class of functions with $2n$ square root-type singularities at the slit corners).

Since the above equations have unique solutions [7], Eq. (7) and conditions (8) also have a solution for any arbitrary value of the right-hand side. When $n = 1$, Eq. (7) has the obvious solution $F(\zeta) = \zeta$, $\omega(\zeta) = ba^{-1}\zeta^{-1}$. The function $\omega_0(\zeta) = C(\zeta + ba^{-1}\zeta^{-1})$ coincides with one obtained in [2] if in addition we map the outside of the unit circle on the outside of the segment [−1, 1] using the Joukowski function

$$v(\zeta) = {}^1/_2 (\zeta + \zeta^{-1})$$

When $n > 1$, we can solve (7) using the method of least squares in $L_2(\Gamma)$ and, since Γ is composed of circumferences, use the finite-dimensional approximations of the form

$$\sum_{p=1}^{m} \left\{ \frac{\alpha_{pk}}{(t - a_k)^p} + \frac{\beta_{pk}}{(\bar{t} - \bar{a}_k)^p} \right\} + \int_{\Gamma_k} f(t)\, dS, \quad t \in \Gamma_k$$

($f(t)$ represents the right-hand side of (7)). Let us number the unknowns α_{pk} and β_{pk} according to the rule

$$x_{2(p-1)n+2k-1} = \alpha_{pk}, \quad x_{2(p-1)n+2k} = \beta_{pk}$$

$$p = 1, 2, \ldots, m, \quad k = 1, 2, \ldots, n$$

$\alpha_{pk}, \beta_{pk} = x_i$ and $\alpha_{ul}, \beta_{ul} = x_j$ we obtain

$$a_{ij} = \frac{1}{2\pi} \int_\Gamma \frac{dS}{(t - a_k)^p (t - a_l)^u} = \sum_{s=1}^{n}{}' r_s (a_s - a_k)^{-1} (a_s - a_l)^{-1} + \tag{9}$$

$$\begin{cases} 0, & k = l \\ (-1)^p \binom{p}{p+u-1} \dfrac{pr_k + ur_l}{(a_k - a_l)^{p+u}}, & k \neq l \end{cases}$$

when $i + j - 1$ is even, and

$$a_{ij} = \frac{1}{2\pi} \int_\Gamma \frac{dS}{(t - a_k)^p (\bar{t} - \bar{a}_l)^u} = \sum_{s=1}^{n}{}' (-1)^u r_s \sum_{q=1}^{u} \binom{q-1}{u-1} \times \tag{10}$$

$$\binom{p-1}{p+u-q-1} \frac{(r_s^2 / (\bar{a}_s - \bar{a}_l))^{u-q} (a_l - a_s)^{p-q}}{(r_s^2 - (a_k - a_s)(\bar{a}_s - \bar{a}_l))^{p+u-q}} + r_k^{-l}$$

when $i + j - 1$ is odd.

The integrals in (9) and (10) can be evaluated using the method of residues. A prime means that the terms containing $s = k, l$ are omitted from the corresponding sum, and $\{a_{ij}\}$ is a symmetric matrix of the normal system. The expressions for the free term of analogous, though simpler structure, have been omitted to save space.

A program for solving the system using the square root method was written and used to perform the calculations on a computer. In practice, not only the number of holes is given, but also their relative sizes and distribution, i.e. the geometry of the region. The latter depends only implicitly on the input parameters a_k and r_k of the program.

From the number of runs accumulated we can assert that, if the holes are not too near each other, then the parameters should be chosen by mapping the given geometry of the z-plane onto the ζ-plane, otherwise choose one of the several computational versions available.

The method converges rapidly irrespective to the relative distribution of the holes.

Let us specify the load for n cyclically and symmetrically distributed holes as follows: $\sigma_r = p$, $\tau_{r\theta} = 0$, $\sigma_x, \sigma_y, \tau_{xy} = 0$ (uniform pressure at the hole contours and zero load at infinity), $\tau = 2\pi/n$, $|a_k| = H$. We have

$$a_{k+1} = e^{i\tau} a_k, \quad r_k = r, \quad k = 1, 2, \ldots, n$$

$$\omega(e^{i\tau}\zeta) = e^{i\tau}\omega(\zeta), \quad F(e^{i\tau}\zeta) = e^{-i\tau} F(\zeta)$$

therefore

$$\alpha_{pk} = e^{-i\tau}\alpha_{p,\,k-1}, \quad \beta_{pk} = e^{i\tau}\beta_{p,\,k-1}$$

and for a fixed m the order of the system (9) reduces to $2m$. In the course of solving (9) we varied m from 3 to 15. Figure 1 depicts the required curves for $n = 2$ (a), $n = 3$ (b), $n = 4$ (c) and $n = 6$ (d). The curves $1 - 4$ correspond to the values of the single-valued independent parameter $\lambda = Hr^{-1} \sin \tau / 2$, $\lambda > 1$, equal to 1.01, 1.1, 1.3 and 1.5. When $\lambda \gg 1$, Eq.(7) with the given boundary conditions, has the obvious solution

$$F(\zeta) = \sum_{k=1}^{n} (a_k - \zeta)^{-1}, \quad \overline{\omega(\zeta)} = \sum_{k=1}^{n}{}' (\zeta - \alpha_k)^{-1}, \quad \zeta \in \Gamma_l \tag{11}$$

The prime denotes that the term with $k = l$ is omitted in the sum. Expanding (11) in a λ-series, we arrive to the solution of the problem which was obtained in [5] by a more complicated manner.

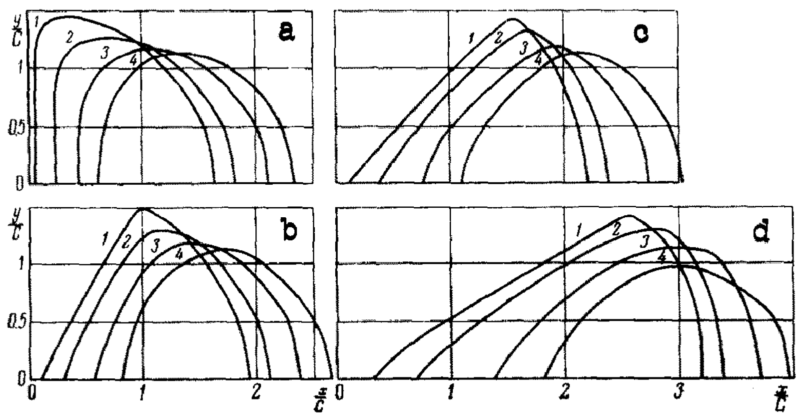

Fig. 1

Note. In [2] the load is restricted to $1 < b / a < -0.187$ for $n = 2$. This results in a serious arithmetical error. The formula (3.17) in [2] should have the form $x_B = C_1 (1.23 + 0.23\, b / a)$, i.e. $|b / a| < 1$, and as the result, the case $b = 0$ is now included in the problem under discussion. The length of the slits and the distance between them should not be fixed for the symmetric case when $n = 2$ since this reduces the generality, but the admissible values of λ for the chosen ratio b / a must be found from the explicit definition of the curve given by (3.17) and (3.18) in [2].

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MAXIMUM PRINCIPLE IN PROBLEMS OF OPTIMAL DESIGN OF REINFORCED SHELLS FOR NONUNIFORM LOADING

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We use the Pontriagin maximum principle to solve the problem of weight-optimal reinforcement of a shell acted upon by a nonuniform axisymmetric external load. When the problems of optimizing the constructional parameters and restrictions are formulated, a class of solutions is always indicated and the optimal solution is chosen from this class. Earlier, the authors of [1] used the Pontriagin maximum principle to obtain the optimal distribution of material along the length of the shell under a nonuniform load. Below we solve a similar problem with a preliminary condition that the shell has constant thickness and transverse reinforcing supports.

We consider a semimembrane model of the shell, in which the axis of the frame is assumed to coincide with the median surface, and be inextensible. After separating the variables, the equation of stability yields

$$\frac{d^4\varphi_n}{dx^4} - \alpha_n^4\varphi_n = 0, \quad \alpha_n^4 = \frac{q(x)R}{E\delta} n^4(n^2-1) - \frac{D}{E\delta R^2} n^4(n^2-1)^2 \tag{1}$$

The conditions of compatibility of deformations must hold at the points of the frame supports $x = l_1, l_2, \ldots, l_m$. Taking into account the fact that a passage across the frame is accompanied by a jump in the shearing and the longitudinal forces, we obtain the relations connecting the stresses and displacements in the form

$$\varphi_+ = \varphi_-, \quad \varphi_+' = \varphi_-', \quad \varphi_+'' = \varphi_-'' + \gamma_2\varphi', \quad \varphi_+''' = \varphi_-''' - \gamma_1\varphi \tag{2}$$

$$\gamma_1 = \frac{n^4(n^2-1)}{E\delta R}\left[\frac{E^\circ I_x(n^2-1)}{R^2} - N^\circ\right], \quad \gamma_2 = \frac{n^2(n^2-1)}{E\delta R^3}\left(\frac{n^2}{E^\circ I_z} + \frac{1}{GI_*}\right)^{-1}$$

Here and henceforth we adopt the following notation: l, R and δ are the length, radius